## Riemann Sum FRQ

## The Riemann Sum (common concepts worked in on the AP test...this is not an all-inclusive list of topics)

$1^{\text {st }}$ ) Notice the units of the function you are starting with!!
'Amount Function' vs 'Rate of Change function'

| $t$ <br> (years) | 2 | 3 | 5 | 7 | 10 |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $H(t)$ <br> $($ meters $)$ | 1.5 | 2 | 6 | 11 | 15 |$\quad$| $t$ <br> (hours) | 0 | 1 | 3 | 6 | 8 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $R(t)$ <br> (liters $/$ hour) | 1340 | 1190 | 950 | 740 | 700 |

How can you tell which type of function you are working with in the Data?

Find the derivative at a value and state the derivative's meaning (make sense of the unit!!)
a) Do you need to demonstrate work? If so, how?
b) Find $\mathrm{H}^{\prime}(6)$ and state the meaning of this value (includes indicated units of measure)
c) Find $\mathrm{R}^{\prime}(7)$ and state the meaning of this value (includes indicating units of measure)

Finding the Riemann sum (what does it tell you...UNITS?!)
Approximate the value of $\int_{0}^{8} R(t) d t$
a) with a right riemann sum
b) with a left riemann sum
c) with a trapezoidal sum

## Riemann Sum FRQ

Approximate the value of $\int_{2}^{10} H(t) d t$
a) with a right riemann sum
b) with a left riemann sum
c) with a trapezoidal sum
d) How would the table look different if they asked for a midpoint sum?

Special topic related to the different function ( $\mathrm{fvs} \mathrm{f}^{\prime}$ ) that is often asked
'Amount Function' vs 'Rate of Change function'
$\frac{1}{8} \int_{2}^{10} H(t) d t$
$\int_{0}^{8}|R(t)| d t$
'Amount Function'

| $t$ <br> (years) | 2 | 3 | 5 | 7 | 10 |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $H(t)$ <br> (meters) | 1.5 | 2 | 6 | 11 | 15 |

vs
'Rate of Change function'

| $t$ <br> (hours) | 0 | 1 | 3 | 6 | 8 |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $R(t)$ <br> (liters / hour) | 1340 | 1190 | 950 | 740 | 700 |

Is the estimate determined by the riemann sum an over or under estimate?
a) what must be stated in the problem in order for this to be asked?
b) Is the right reimann sum of $\frac{1}{8} \int_{2}^{10} H(t) d t$ an over or under estimate and explain your reasoning
c) Is the left reimann sum of $\frac{1}{8} \int_{2}^{10} H(t) d t$ an over or under estimate and explain your reasoning
d) Is the right reimann sum of $\int_{0}^{8} R(t) d t$ an over or under estimate and explain your reasoning
e) Is the left reimann sum of $\int_{0}^{8} R(t) d t$ an over or under estimate and explain your reasoning

The average velocity (could ask for $f$ or $\mathrm{f}^{\prime}$, requires different calculus)
'Amount Function' vs 'Rate of Change function'

## Fundamental theorem of calculus (usually with f...but could be asked regarding f') (2011c)

$\int_{2}^{10} H^{\prime}(t) d t \quad \int_{0}^{8} R^{\prime \prime}(t) d t$

Mean Value Theorem and/or Intermediate Value Theorem
a) What must be stated in problem in order to use the Intermediate Value Theorem?
b) What must be stated in the problem in order to use the Mean Value Theorem?

## Using R(t)

c) Does the data support the conclusion that $R(t)=1000$ liters per minute at some time $t$ with $0<t<3$. Give a reason for you answer
d) Is there a time $t, 0<t<8$, at which $R^{\prime}(t)=120$ ? Justify your answer
(2 methods we can use...we will demonstrate both)

## Using $\mathrm{H}(\mathrm{t})$

e) Does the data support the conclusion that $\mathrm{H}(\mathrm{t})=13$ liters per minute at some time t with $2<\mathrm{t}<10$. Give a reason for you answer
f) Is there a time $\mathrm{t}, 2<\mathrm{t}<10$, at which $\mathrm{H}^{\prime}(\mathrm{t})=2.5$ ? Justify your answer

| $t$ <br> (hours) | 0 | 0.3 | 1.7 | 2.8 | 4 |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $v_{P}(t)$ <br> (meters per hour) | 0 | 55 | -29 | 55 | 48 |

2. The velocity of a particle, $P$, moving along the $x$-axis is given by the differentiable function $v_{P}$, where $v_{P}(t)$ is measured in meters per hour and $t$ is measured in hours. Selected values of $v_{P}(t)$ are shown in the table above. Particle $P$ is at the origin at time $t=0$.
(a) Justify why there must be at least one time $t$, for $0.3 \leq t \leq 2.8$, at which $v_{P}{ }^{\prime}(t)$, the acceleration of particle $P$, equals 0 meters per hour per hour.
(b) Use a trapezoidal sum with the three subintervals $[0,0.3],[0.3,1.7]$, and $[1.7,2.8]$ to approximate the value of $\int_{0}^{2.8} v_{P}(t) d t$.
(c) A second particle, $Q$, also moves along the $x$-axis so that its velocity for $0 \leq t \leq 4$ is given by $v_{Q}(t)=45 \sqrt{t} \cos \left(0.063 t^{2}\right)$ meters per hour. Find the time interval during which the velocity of particle $Q$ is at least 60 meters per hour. Find the distance traveled by particle $Q$ during the interval when the velocity of particle $Q$ is at least 60 meters per hour.
(d) At time $t=0$, particle $Q$ is at position $x=-90$. Using the result from part (b) and the function $v_{Q}$ from part (c), approximate the distance between particles $P$ and $Q$ at time $t=2.8$.

| $t$ <br> (years) | 2 | 3 | 5 | 7 | 10 |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $H(t)$ <br> (meters) | 1.5 | 2 | 6 | 11 | 15 |

4. The height of a tree at time $t$ is given by a twice-differentiable function $H$, where $H(t)$ is measured in meters and $t$ is measured in years. Selected values of $H(t)$ are given in the table above.
(a) Use the data in the table to estimate $H^{\prime}(6)$. Using correct units, interpret the meaning of $H^{\prime}(6)$ in the context of the problem.
(b) Explain why there must be at least one time $t$, for $2<t<10$, such that $H^{\prime}(t)=2$.
(c) Use a trapezoidal sum with the four subintervals indicated by the data in the table to approximate the average height of the tree over the time interval $2 \leq t \leq 10$.
(d) The height of the tree, in meters, can also be modeled by the function $G$, given by $G(x)=\frac{100 x}{1+x}$, where $x$ is the diameter of the base of the tree, in meters. When the tree is 50 meters tall, the diameter of the base of the tree is increasing at a rate of 0.03 meter per year. According to this model, what is the rate of change of the height of the tree with respect to time, in meters per year, at the time when the tree is 50 meters tall?

| $h$ <br> (feet) | 0 | 2 | 5 | 10 |
| :---: | :---: | :---: | :---: | :---: |
| $A(h)$ <br> (square feet) | 50.3 | 14.4 | 6.5 | 2.9 |

1. A tank has a height of 10 feet. The area of the horizontal cross section of the tank at height $h$ feet is given by the function $A$, where $A(h)$ is measured in square feet. The function $A$ is continuous and decreases as $h$ increases. Selected values for $A(h)$ are given in the table above.
(a) Use a left Riemann sum with the three subintervals indicated by the data in the table to approximate the volume of the tank. Indicate units of measure.
(b) Does the approximation in part (a) overestimate or underestimate the volume of the tank? Explain your reasoning.
(c) The area, in square feet, of the horizontal cross section at height $h$ feet is modeled by the function $f$ given by $f(h)=\frac{50.3}{e^{0.2 h}+h}$. Based on this model, find the volume of the tank. Indicate units of measure.
(d) Water is pumped into the tank. When the height of the water is 5 feet, the height is increasing at the rate of 0.26 foot per minute. Using the model from part (c), find the rate at which the volume of water is changing with respect to time when the height of the water is 5 feet. Indicate units of measure.

# 2016 AP ${ }^{\circledR}$ CALCULUS AB FREE-RESPONSE QUESTIONS 

## CALCULUS AB

SECTION II, Part A
Time- $\mathbf{3 0}$ minutes
Number of problems-2
A graphing calculator is required for these problems.

| $t$ <br> (hours) | 0 | 1 | 3 | 6 | 8 |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $R(t)$ <br> (liters / hour) | 1340 | 1190 | 950 | 740 | 700 |

1. Water is pumped into a tank at a rate modeled by $W(t)=2000 e^{-t^{2} / 20}$ liters per hour for $0 \leq t \leq 8$, where $t$ is measured in hours. Water is removed from the tank at a rate modeled by $R(t)$ liters per hour, where $R$ is differentiable and decreasing on $0 \leq t \leq 8$. Selected values of $R(t)$ are shown in the table above. At time $t=0$, there are 50,000 liters of water in the tank.
(a) Estimate $R^{\prime}(2)$. Show the work that leads to your answer. Indicate units of measure.
(b) Use a left Riemann sum with the four subintervals indicated by the table to estimate the total amount of water removed from the tank during the 8 hours. Is this an overestimate or an underestimate of the total amount of water removed? Give a reason for your answer.
(c) Use your answer from part (b) to find an estimate of the total amount of water in the tank, to the nearest liter, at the end of 8 hours.
(d) For $0 \leq t \leq 8$, is there a time $t$ when the rate at which water is pumped into the tank is the same as the rate at which water is removed from the tank? Explain why or why not.

## 2015 AP $^{\oplus}$ CALCULUS AB FREE-RESPONSE QUESTIONS

# CALCULUS AB <br> SECTION II, Part B 

Time-60 minutes
Number of problems-4

No calculator is allowed for these problems.

| $t$ <br> (minutes) | 0 | 12 | 20 | 24 | 40 |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $v(t)$ | 0 | 200 | 240 | -220 | 150 |

3. Johanna jogs along a straight path. For $0 \leq t \leq 40$, Johanna's velocity is given by a differentiable function $v$. Selected values of $v(t)$, where $t$ is measured in minutes and $v(t)$ is measured in meters per minute, are given in the table above.
(a) Use the data in the table to estimate the value of $v^{\prime}(16)$.
(b) Using correct units, explain the meaning of the definite integral $\int_{0}^{40}|v(t)| d t$ in the context of the problem. Approximate the value of $\int_{0}^{40}|v(t)| d t$ using a right Riemann sum with the four subintervals indicated in the table.
(c) Bob is riding his bicycle along the same path. For $0 \leq t \leq 10$, Bob's velocity is modeled by $B(t)=t^{3}-6 t^{2}+300$, where $t$ is measured in minutes and $B(t)$ is measured in meters per minute. Find Bob's acceleration at time $t=5$.
(d) Based on the model $B$ from part (c), find Bob's average velocity during the interval $0 \leq t \leq 10$.

## 2014 AP ${ }^{\circledR}$ CALCULUS AB FREE-RESPONSE QUESTIONS

| $t$ <br> (minutes) | 0 | 2 | 5 | 8 | 12 |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $v_{A}(t)$ | 0 | 100 | 40 | -120 | -150 |
| (meters/minute) |  |  |  |  |  |

4. Train $A$ runs back and forth on an east-west section of railroad track. Train A's velocity, measured in meters per minute, is given by a differentiable function $v_{A}(t)$, where time $t$ is measured in minutes. Selected values for $v_{A}(t)$ are given in the table above.
(a) Find the average acceleration of train $A$ over the interval $2 \leq t \leq 8$.
(b) Do the data in the table support the conclusion that train $A$ 's velocity is -100 meters per minute at some time $t$ with $5<t<8$ ? Give a reason for your answer.
(c) At time $t=2$, train $A$ 's position is 300 meters east of the Origin Station, and the train is moving to the east. Write an expression involving an integral that gives the position of train $A$, in meters from the Origin Station, at time $t=12$. Use a trapezoidal sum with three subintervals indicated by the table to approximate the position of the train at time $t=12$.
(d) A second train, train $B$, travels north from the Origin Station. At time $t$ the velocity of train $B$ is given by $v_{B}(t)=-5 t^{2}+60 t+25$, and at time $t=2$ the train is 400 meters north of the station. Find the rate, in meters per minute, at which the distance between train $A$ and train $B$ is changing at time $t=2$.

# CALCULUS AB <br> SECTION II, Part B <br> Time- $\mathbf{6 0}$ minutes <br> Number of problems-4 

## No calculator is allowed for these problems.

| $t$ <br> (minutes) | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $C(t)$ <br> (ounces) | 0 | 5.3 | 8.8 | 11.2 | 12.8 | 13.8 | 14.5 |

3. Hot water is dripping through a coffeemaker, filling a large cup with coffee. The amount of coffee in the cup at time $t, 0 \leq t \leq 6$, is given by a differentiable function $C$, where $t$ is measured in minutes. Selected values of $C(t)$, measured in ounces, are given in the table above.
(a) Use the data in the table to approximate $C^{\prime}(3.5)$. Show the computations that lead to your answer, and indicate units of measure.
(b) Is there a time $t, 2 \leq t \leq 4$, at which $C^{\prime}(t)=2$ ? Justify your answer.
(c) Use a midpoint sum with three subintervals of equal length indicated by the data in the table to approximate the value of $\frac{1}{6} \int_{0}^{6} C(t) d t$. Using correct units, explain the meaning of $\frac{1}{6} \int_{0}^{6} C(t) d t$ in the context of the problem.
(d) The amount of coffee in the cup, in ounces, is modeled by $B(t)=16-16 e^{-0.4 t}$. Using this model, find the rate at which the amount of coffee in the cup is changing when $t=5$.

## 2012 AP ${ }^{\circledR}$ CALCULUS AB FREE-RESPONSE QUESTIONS

## CALCULUS AB <br> SECTION II, Part A

Time- $\mathbf{3 0}$ minutes
Number of problems-2

## A graphing calculator is required for these problems.

| $t$ (minutes) | 0 | 4 | 9 | 15 | 20 |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $W(t)$ (degrees Fahrenheit) | 55.0 | 57.1 | 61.8 | 67.9 | 71.0 |

1. The temperature of water in a tub at time $t$ is modeled by a strictly increasing, twice-differentiable function $W$, where $W(t)$ is measured in degrees Fahrenheit and $t$ is measured in minutes. At time $t=0$, the temperature of the water is $55^{\circ} \mathrm{F}$. The water is heated for 30 minutes, beginning at time $t=0$. Values of $W(t)$ at selected times $t$ for the first 20 minutes are given in the table above.
(a) Use the data in the table to estimate $W^{\prime}(12)$. Show the computations that lead to your answer. Using correct units, interpret the meaning of your answer in the context of this problem.
(b) Use the data in the table to evaluate $\int_{0}^{20} W^{\prime}(t) d t$. Using correct units, interpret the meaning of $\int_{0}^{20} W^{\prime}(t) d t$ in the context of this problem.
(c) For $0 \leq t \leq 20$, the average temperature of the water in the tub is $\frac{1}{20} \int_{0}^{20} W(t) d t$. Use a left Riemann sum with the four subintervals indicated by the data in the table to approximate $\frac{1}{20} \int_{0}^{20} W(t) d t$. Does this approximation overestimate or underestimate the average temperature of the water over these 20 minutes? Explain your reasoning.
(d) For $20 \leq t \leq 25$, the function $W$ that models the water temperature has first derivative given by $W^{\prime}(t)=0.4 \sqrt{t} \cos (0.06 t)$. Based on the model, what is the temperature of the water at time $t=25$ ?

| $t$ <br> (minutes) | 0 | 2 | 5 | 9 | 10 |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $H(t)$ <br> (degrees Celsius) | 66 | 60 | 52 | 44 | 43 |

2. As a pot of tea cools, the temperature of the tea is modeled by a differentiable function $H$ for $0 \leq t \leq 10$, where time $t$ is measured in minutes and temperature $H(t)$ is measured in degrees Celsius. Values of $H(t)$ at selected values of time $t$ are shown in the table above.
(a) Use the data in the table to approximate the rate at which the temperature of the tea is changing at time $t=3.5$. Show the computations that lead to your answer.
(b) Using correct units, explain the meaning of $\frac{1}{10} \int_{0}^{10} H(t) d t$ in the context of this problem. Use a trapezoidal sum with the four subintervals indicated by the table to estimate $\frac{1}{10} \int_{0}^{10} H(t) d t$.
(c) Evaluate $\int_{0}^{10} H^{\prime}(t) d t$. Using correct units, explain the meaning of the expression in the context of this problem.
(d) At time $t=0$, biscuits with temperature $100^{\circ} \mathrm{C}$ were removed from an oven. The temperature of the biscuits at time $t$ is modeled by a differentiable function $B$ for which it is known that
$B^{\prime}(t)=-13.84 e^{-0.173 t}$. Using the given models, at time $t=10$, how much cooler are the biscuits than the tea?

| $t$ <br> (hours) | 0 | 2 | 5 | 7 | 8 |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $E(t)$ <br> (hundreds of <br> entries) | 0 | 4 | 13 | 21 | 23 |

2. A zoo sponsored a one-day contest to name a new baby elephant. Zoo visitors deposited entries in a special box between noon $(t=0)$ and 8 P.M. $(t=8)$. The number of entries in the box $t$ hours after noon is modeled by a differentiable function $E$ for $0 \leq t \leq 8$. Values of $E(t)$, in hundreds of entries, at various times $t$ are shown in the table above.
(a) Use the data in the table to approximate the rate, in hundreds of entries per hour, at which entries were being deposited at time $t=6$. Show the computations that lead to your answer.
(b) Use a trapezoidal sum with the four subintervals given by the table to approximate the value of $\frac{1}{8} \int_{0}^{8} E(t) d t$. Using correct units, explain the meaning of $\frac{1}{8} \int_{0}^{8} E(t) d t$ in terms of the number of entries.
(c) At 8 P.M., volunteers began to process the entries. They processed the entries at a rate modeled by the function $P$, where $P(t)=t^{3}-30 t^{2}+298 t-976$ hundreds of entries per hour for $8 \leq t \leq 12$. According to the model, how many entries had not yet been processed by midnight $(t=12)$ ?
(d) According to the model from part (c), at what time were the entries being processed most quickly? Justify your answer.
